Recap session 2

Thursday, 28 May 2020 08:12

... Duality continued

$\mathbb{H}^n$, ... OSL $(\mathfrak{g}_0, \theta_0)$, $\quad \mathfrak{g} = \mathfrak{g}_0 \oplus i\cdot\mathfrak{g}_0 = \underline{\mathfrak{k}_0} \oplus \mathfrak{p}_0 \oplus i\cdot\mathfrak{k}_0 \oplus \underline{i\mathfrak{p}_0}$

$\mathfrak{g}^* = \mathfrak{k}_0 \oplus i\cdot\mathfrak{p}_0$, complex conjugation

$$\mathfrak{g}^* = \left\{ \begin{pmatrix} A & iB \\ iB^T & 0 \end{pmatrix} : \begin{array}{l} A \in \mathbb{R}^{n\times n} \\ A^T + A = 0, \quad B \in \mathbb{R}^{1\times n} \end{array} \right\}$$

<u>Claim:</u> $\mathfrak{g}^* \xrightarrow{\sim} \mathfrak{so}(n+1)$ Lie algebra isomorphism.

<u>Proof:</u>
$$\begin{pmatrix} A & iB \\ iB^T & 0 \end{pmatrix} \longmapsto \begin{pmatrix} A & -B \\ B^T & 0 \end{pmatrix} = \begin{pmatrix} Id_n & \\ & -i \end{pmatrix} \begin{pmatrix} A & iB \\ iB^T & 0 \end{pmatrix} \begin{pmatrix} Id_n & \\ & i \end{pmatrix} \qquad \square$$

$$\begin{tikzcd} \mathfrak{g}^* \arrow[r, "\text{complex conjugation}"] \arrow[d, "\wr"] & \mathfrak{g}^* \arrow[d, "\wr"] \\ \mathfrak{so}(n+1) \arrow[r, "\theta^*"] & \mathfrak{so}(n+1) \end{tikzcd}$$

We get dual OSL which is isomorphic to $(\mathfrak{so}(n+1), \theta^*)$, and $\mathfrak{k}^* \cong \mathfrak{so}(n)$

$\cancel{\Rightarrow} \quad (\mathbb{H}^n)^* = SO(n+1)/SO(n) = S^n$

<u>Actually:</u> $G := \mathrm{Aut}(\mathfrak{g}^*)^0 = \mathrm{Aut}(\mathfrak{so}(n+1))^0 \underset{\substack{SO(n+1)\\ \text{connected} \\ \text{semisimple}}}{\cong} \mathrm{Ad}_{SO(n+1)}(SO(n+1))$

$\underset{\substack{SO(n+1)\\ \text{connected}}}{\cong} SO(n+1)/Z(SO(n+1)) = \begin{cases} \cong SO(n+1) & \text{if } n \text{ is even} \\ \cong SO(n+1)/\pm Id & \text{if } n \text{ is odd} \end{cases}$

What is K? ...

$K \cong SO(n)$

$\Rightarrow M^* = G/K = \ldots, \quad S^n$ or $\mathbb{R}P^n$

<u>Note:</u> If M is of non-compact type, then $(M^*)^* = M$

of compact type: $((\mathbb{R}P^2)^*)^* = (\mathbb{H}^2)^* = S^2$

Riemannian Geometry

$\pi: G \to M$, $g \mapsto g\cdot 0$,

$$\begin{tikzcd} \mathfrak{p} \arrow[r, "D_e\pi|_\mathfrak{p}", "\sim"'] \arrow[d, "\exp"] & T_0M \arrow[d, "\mathrm{Exp}_0"] \\ G \arrow[r, "\pi"] & M \end{tikzcd}$$

$X_\mathfrak{p}$

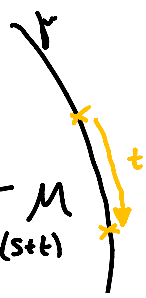
$\mathfrak{g} \xrightarrow{\exp} G$

$\cap$

To prove this we need transvections.

$\tau_t := s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}$

$D\tau_t|_{\gamma(s)} : T_{\gamma(s)}M \to T_{\gamma(s+t)}M$



$\to\ \mathfrak{g} \xrightarrow{\exp} G$

$$\begin{array}{ccc} T_X\mathfrak{g} & \xrightarrow{\partial_X \exp} & T_{\exp(X)}G \\ \downarrow & & \uparrow \wr\ \partial_e L_{\exp(X)} \\ \mathfrak{g} & \longrightarrow & \mathfrak{g} = T_{Id}G \\ Y & \longmapsto & \sum_{n=0}^{\infty} \frac{(-ad(Y))^n}{(n+1)!} \end{array}$$

$\partial_{\gamma(s)}\tau_t : T_{\gamma(s)}M \to T_{\gamma(s+t)}M$

implements the parallel transport.

$\to\ \forall X, Y, Z \in \mathfrak{p}, \Rightarrow \overline{X}, \overline{Y}, \overline{Z} \in T_o M$

$\Rightarrow R(\overline{X}, \overline{Y})\overline{Z} = -\overline{[[X,Y],Z]}$

Sectional curvature:

$\sigma_p(\mathbb{R}v \oplus \mathbb{R}w) := g_p(R(v,w)v, w)$

$\to$ If M is compact type, then $\sigma_p \geq 0$

M is non-compact type, then $\sigma_p \leq 0$

Euclidean type, then $\sigma_p = 0$.

$\to$ If M is of compact type, then: M is compact

$Is(M)^\circ$ is compact..

$\to$ If M is of non-compact type, then:

- $\mathrm{Exp}_o : T_o M \to M$ is a diffeomorphism.

  $\Rightarrow M$ is simply connected.

- $\mathfrak{p} \times K \to G$ is a diffeomorphism.

  $(X, k) \longmapsto \exp(X)\cdot k$

- M is $CAT(0)$ space.

Lie triple systems.

Definition 2.44. Let $\mathfrak{g}$ be a Lie algebra. A subset $\mathfrak{n} \subseteq \mathfrak{g}$ is a *Lie triple system* if $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] \subseteq \mathfrak{n}$.

Lie triple systems correspond to totally geodesic subspaces.

Special case: If $\mathfrak{n}$ Euclidean, $[\mathfrak{n},\mathfrak{n}]=0$, $\mathfrak{n} \subset \mathfrak{p}$, $\boxed{\mathrm{Exp}_o(\partial_e \pi(\mathfrak{n})) \text{ is a flat.}}$

Ex: $SL(n,\mathbb{R})/SO(n)$, $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$, $\Theta(X) = -X^T$,

$n=2$: $\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{so}(2) \oplus \{X : X^T = X\}$

$n=2$: $\quad \mathfrak{sl}(2,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{so}(2) \oplus \{X : X^T = X\}$

$$= \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle \oplus \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$$

$\mathfrak{a} := \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$ Euclidean, abelian subspace of $\mathfrak{p}$.

Let $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, let $\mathfrak{a}'$ be another abelian subspace $\mathfrak{a} < \mathfrak{a}' < \mathfrak{p}$

$$\mathfrak{a}' < Z_{\mathfrak{g}}(H) \cap \mathfrak{p} = \{X \in \mathfrak{p} : [X,H] = 0\} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle = \mathfrak{a}.$$

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \neq 0$$

$\Rightarrow$ $\mathfrak{a}$ is maximal abelian subspace.

Def: Rank of $\mathfrak{g}/M$ is the dimension of the maximal abelian subspace / flat.

$\Rightarrow n=2$: $\; SL(2,\mathbb{R})/SO(2)$ has rank 1.

For $SL(n,\mathbb{R})/SO(n)$ we consider $\mathfrak{a} = \left\langle \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & \ddots \; 0 \end{pmatrix}, \dots, \begin{pmatrix} \ddots & & \\ & 0 & \\ & & 1 & \\ & & & -1 \end{pmatrix} \right\rangle$

$H = \begin{pmatrix} -\frac{n}{2} & & \\ & \ddots & \\ & & \frac{n}{2} \end{pmatrix}$, $\quad Z_{\mathfrak{g}}(H) \cap \mathfrak{p} = \mathfrak{a} \;\Rightarrow\; \mathfrak{a}$ is maximal.

$$\Rightarrow \operatorname{rank}\left(SL(n,\mathbb{R})/SO(n)\right) = n-1$$

of non-compact type.

$M = M_0 \times M_- \times M_+$

$\operatorname{rank}(M_0) = \dim(M_0) = n, \quad M_0 = \mathbb{R}^n$

Remark: $\operatorname{rank}(\mathbb{H}^n) = 1$, $\operatorname{rank}\left(SL(n,\mathbb{R})/SO(n)\right) \neq 1$ for $n \geq 3$.

$$O(n,1)^0/SO(n) = \mathbb{H}^n \neq SL(n,\mathbb{R})/SO(n) = P_1(n)$$

Root spaces

Ex: $\mathfrak{sl}(2,\mathbb{R})$, $\; \mathfrak{a} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$ maximal flat.

$\operatorname{ad}(H): \mathfrak{g} \to \mathfrak{g}$, $\quad X \mapsto [H,X]$,

$$\operatorname{ad}(H)\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & -a \end{pmatrix} - \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} a & b \\ -c & a \end{pmatrix} - \begin{pmatrix} a & -b \\ c & a \end{pmatrix} = \begin{pmatrix} 0 & 2b \\ -2c & 0 \end{pmatrix}$$

$E_1(\operatorname{ad}(H)) = \left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle = \mathfrak{g}_2 = \{X \in \mathfrak{g} : \operatorname{ad}(H) X = 2 \cdot X\}$

$$E_2(ad(H)) = \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rangle = \mathfrak{g}_2 = \{ X \in \mathfrak{g} : ad(H)X = 2 \cdot X \}$$

$$E_{-2}(ad(H)) = \langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rangle \qquad \Rightarrow \quad \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2} = \mathfrak{a} \oplus \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rangle$$

$$E_0(ad(H)) = \mathfrak{a} \qquad \text{Roots: are } 2, -2,$$

$n=3$: $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$: $\mathfrak{a} = \langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle =: \langle H_1, H_2 \rangle$

$$ad(H_1)\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} - \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

$a+e+i=0$

$$= \begin{pmatrix} a & b & c \\ -d & -e & -f \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} a & -b & 0 \\ d & -e & 0 \\ g & -h & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2b & c \\ -2d & 0 & -f \\ -g & h & 0 \end{pmatrix}$$

$\mathfrak{g}_\alpha =$

$\mathfrak{g}_\lambda$	$\begin{pmatrix} 0&1&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&1 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0 \\ 1&0&0 \\ 0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0 \\ 0&0&1 \\ 0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&1&0 \end{pmatrix}$	$\mathfrak{a}$
$ad(H_1)$	2	1	-2	-1	-1	1	0
$ad(H_2)$	-1	1	1	2	-1	-2	0
Roots:	α	$\alpha+\beta$	$-\alpha$	β	$-\alpha-\beta$	$-\beta$	0 not a root.

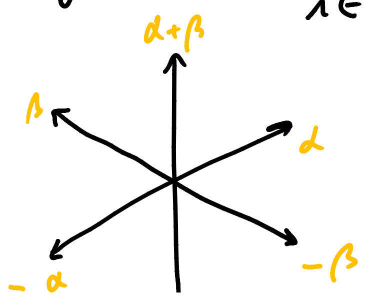
Root: $\alpha \in \mathfrak{a}^*$: $\alpha : \mathfrak{a} \to \mathbb{R}$, $H_1 \mapsto 2$, $H_2 \mapsto -1$; $\beta : \mathfrak{a} \to \mathbb{R}$, $H_1 \mapsto -1$, $H_2 \mapsto 2$

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{sl}(n,\mathbb{R}) : \forall H \in \mathfrak{a} : [H, X] = \alpha(H) \cdot X \}$$

$$[H_1, X] = 2 \cdot X$$

The set of roots $= \Sigma = \{ \alpha, \beta, \alpha+\beta, -\alpha, -\beta, -\alpha-\beta \}$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Sigma} \mathfrak{g}_\lambda = \mathfrak{a} \oplus \langle \begin{pmatrix} 0&1&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, \dots, \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&1&0 \end{pmatrix} \rangle$$



Classification

Irreducible simply connected RSS of compact type :

A) K is a compact simple Lie group. $K \times K / \Delta \cong K$ RSS.

B)

$\left(SL(n,\mathbb{R}) / SO(n) \right)^*$

Label	G	K	Dimension	Rank	Geometric interpretation
AI	$\mathrm{SU}(n)$	$\mathrm{SO}(n)$	$(n-1)(n+2)/2$	$n-1$	Space of real structures on $\mathbb{C}^n$ which leave the complex determinant invariant
AII	$\mathrm{SU}(2n)$	$\mathrm{Sp}(n)$	$(n-1)(2n+1)$	$n-1$	Space of quaternionic structures on $\mathbb{C}^{2n}$ compatible with the Hermitian metric
AIII	$\mathrm{SU}(p+q)$	$\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$	$2pq$	min(p,q)	Grassmannian of complex p-dimensional subspaces of $\mathbb{C}^{p+q}$
BDI	$\mathrm{SO}(p+q)$	$\mathrm{SO}(p)\times\mathrm{SO}(q)$	pq	min(p,q)	Grassmannian of oriented real p-dimensional subspaces of $\mathbb{R}^{p+q}$
DIII	$\mathrm{SO}(2n)$	$\mathrm{U}(n)$	$n(n-1)$	[n/2]	Space of orthogonal complex structures on $\mathbb{R}^{2n}$
CI	$\mathrm{Sp}(n)$	$\mathrm{U}(n)$	$n(n+1)$	n	Space of complex structures on $\mathbb{H}^n$ compatible with the inner product
CII	$\mathrm{Sp}(p+q)$	$\mathrm{Sp}(p)\times\mathrm{Sp}(q)$	$4pq$	min(p,q)	Grassmannian of quaternionic p-dimensional subspaces of $\mathbb{H}^{p+q}$
EI	E_6	$\mathrm{Sp}(4)/\{\pm I\}$	42	6	
EII	E_6	$\mathrm{SU}(6)\cdot\mathrm{SU}(2)$	40	4	Space of symmetric subspaces of $(\mathbb{C}\otimes\mathbb{O})P^2$ isometric to $(\mathbb{C}\otimes\mathbb{H})P^2$
EIII	E_6	$\mathrm{SO}(10)\cdot\mathrm{SO}(2)$	32	2	Complexified Cayley projective plane $(\mathbb{C}\otimes\mathbb{O})P^2$
EIV	E_6	F_4	26	2	Space of symmetric subspaces of $(\mathbb{C}\otimes\mathbb{O})P^2$ isometric to $\mathbb{O}P^2$
EV	E_7	$\mathrm{SU}(8)/\{\pm I\}$	70	7	
EVI	E_7	$\mathrm{SO}(12)\cdot\mathrm{SU}(2)$	64	4	Rosenfeld projective plane $(\mathbb{H}\otimes\mathbb{O})P^2$ over $\mathbb{H}\otimes\mathbb{O}$
EVII	E_7	$E_6\cdot\mathrm{SO}(2)$	54	3	Space of symmetric subspaces of $(\mathbb{H}\otimes\mathbb{O})P^2$ isomorphic to $(\mathbb{C}\otimes\mathbb{O})P^2$
EVIII	E_8	$\mathrm{Spin}(16)/\{\pm\mathrm{vol}\}$	128	8	Rosenfeld projective plane $(\mathbb{O}\otimes\mathbb{O})P^2$
EIX	E_8	$E_7\cdot\mathrm{SU}(2)$	112	4	Space of symmetric subspaces of $(\mathbb{O}\otimes\mathbb{O})P^2$ isomorphic to $(\mathbb{H}\otimes\mathbb{O})P^2$
FI	F_4	$\mathrm{Sp}(3)\cdot\mathrm{SU}(2)$	28	4	Space of symmetric subspaces of $\mathbb{O}P^2$ isomorphic to $\mathbb{H}P^2$
FII	F_4	$\mathrm{Spin}(9)$	16	1	Cayley projective plane $\mathbb{O}P^2$
G	G_2	$\mathrm{SO}(4)$	8	2	Space of subalgebras of the octonion algebra $\mathbb{O}$ which are isomorphic to the quaternion algebra $\mathbb{H}$